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THEORY OF INTERACTION OF GRAVITY WAVES WITH HYDRODYNAMIC TURBULENCE

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1. One of the practical problems in atmospheric and ocean dynamics is the study of the effect of vortex turbulence on the propagation of various types of waves (surface waves, sound, internal waves, etc.). Besides, it is necessary to differentiate the strictly hydrodynamic turbulence from vortex motions accompanying waves [1, 2]. The numerous aspects of the interaction of sound with turbulence have been analyzed in fairly great detail so far (see [3-5] and the literature cited in them). The study of the vortex-wave turbulence with reference to the interaction of gravity waves with vortices is continued in this paper. It is worth emphasizing that unlike [3-5], the present problem has a number of special features associated with the fact that surface waves in deep water are dispersive, the phase velocity v_Φ is a function of wavelength λ: $v_\Phi = \sqrt{g\lambda}$ (g is the acceleration of gravity, $\lambda = \lambda/2\pi$). The logarithmic decrement for gravity waves propagating at the surface of a turbulent liquid has been found on the basis of computed matrix of interaction coefficients. An estimate of the characteristic phase correlation time and the time for making the wave field isotropic in elastic scattering have been obtained. The diffusion approximation has been used to analyze the effect of inelastic scattering on the development of isotropic wave packets as a function of frequency. The limits of the applicability of kinetic equations to describe the mutual interaction of gravity waves in a turbulent medium have been explained. All these problems have been analyzed within the framework of a single formal scheme, viz., the Wyld diagram technique [6].

2. Consider the motion of an incompressible fluid with a free surface of infinite depth. We choose a coordinate system with the z axis vertically up. Let the surface shape be given by the function $z = \eta(\mathbf{r}_\perp, t)$ with the normal $\mathbf{n} = [1 + (\nabla_\perp\eta)^2]^{-1/2}\times(-\nabla_\perp\eta, 1)$.

The kinematic boundary condition

$$\partial\eta/\partial t + (\mathbf{u}\nabla_\perp)\eta = u_z \tag{2.1}$$

is satisfied at the free surface. It couples η to the fluid velocity u, which is governed by the equations

$$\partial\mathbf{u}/\partial t + (\mathbf{u}\nabla)\mathbf{u} = -(1/\rho)\nabla p + \mathbf{g}; \tag{2.2}$$

$$\operatorname{div}\mathbf{u} = 0. \tag{2.3}$$

Here p is the pressure, ρ is the density, g is the acceleration of gravity. The system of equations (2.1)-(2.3) is completed by the dynamic boundary condition at the free surface*

$$p|_{z=\eta} = 0 \tag{2.4}$$

(in which a constant atmospheric pressure is taken as the reference value) and the condition of boundedness of fields as $z \to -\infty$.

In such a system there are, in general, two types of motions: vortex motion (hydrodynamic turbulence) and potential motion (surface waves). Hence it is convenient to divide the velocity field into two parts:

$$\mathbf{u} = \mathbf{u}^l + \mathbf{u}^t,\ \operatorname{rot}\mathbf{u}^l = 0,\ \operatorname{div}\mathbf{u}^t = 0, \tag{2.5}$$

*Surface tension has been neglected here and in what follows.

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in which $\mathbf{u}^l$ can be expressed in the form $\mathbf{u}^l = \nabla\varphi$.

Substituting (2.5) in the system (2.2), (2.3), we obtain the following equations for the vortex and potential components respectively:

$$\partial \mathbf{u}^t/\partial t + (\mathbf{u}^t\nabla)\mathbf{u}^t = -\nabla H - (\mathbf{u}^t\nabla)\nabla\varphi - (\nabla\varphi\nabla)\mathbf{u}^t; \tag{2.6}$$

$$H = \partial\varphi/\partial t + p/\rho + gz + (1/2)(\nabla\varphi)^2; \tag{2.7}$$

$$\Delta\varphi = 0, \ -\infty < z < \eta, \ \varphi \to 0 \text{ for } z \to -\infty. \tag{2.8}$$

Eliminating pressure from Eq. (2.7) and substituting the corresponding expressions in the dynamic boundary conditions (2.4), we find

$$\partial\varphi/\partial t + g\eta = -(1/2)(\nabla\varphi)^2 + H|_{z=\eta}. \tag{2.9}$$

It is convenient to introduce velocity potential at the free surface $\psi(\mathbf{r}_\perp, t) = \varphi(\mathbf{r}_\perp, \eta(\mathbf{r}_\perp, t), t)$, as the reference variable. As shown in [7], for purely potential motion the quantities $\psi(\mathbf{r}_\perp, t)$ and $\eta(\mathbf{r}_\perp, t)$ form a canonical pair.*

The expressions for $\psi(\mathbf{r}_\perp, t)$ follows from (2.9) using (2.1)

$$\frac{\partial\psi}{\partial t} + g\eta = -\frac{1}{2}(\nabla_\perp\varphi)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial z}\right)^2 - (\mathbf{u}\nabla_\perp)\eta\frac{\partial\varphi}{\partial z} + u_z^t\frac{\partial\varphi}{\partial z} + H|_{z=\eta}. \tag{2.10}$$

In order to determine the function H we take the divergence of Eq. (2.6) and using the condition $\operatorname{div}\mathbf{u}^t = 0$, we get for H

$$\Delta H = -\operatorname{div}\mathbf{R} \equiv -\chi, \tag{2.11}$$

where $\mathbf{R} = (\mathbf{u}^t\nabla)\mathbf{u}^t + (\nabla\varphi\nabla)\mathbf{u}^t + (\mathbf{u}^t\nabla)\nabla\varphi$.

Equation (2.11) is transformed to the Fourier form in terms of the transverse coordinates

$$H(\mathbf{r}_\perp, z, t) = \int H_\mathbf{k}(z, t)\, e^{i\mathbf{k}\mathbf{r}_\perp} d\mathbf{k}, \quad \chi(\mathbf{r}_\perp, z, t) = \int \chi_\mathbf{k}(z, t)\, e^{i\mathbf{k}\mathbf{r}_\perp} d\mathbf{k}.$$

We have

$$(\partial^2/\partial z^2 - k^2)H_\mathbf{k} = -\chi_\mathbf{k}. \tag{2.12}$$

The general solution of the nonhomogeneous equation (2.12) is expressed as follows:

$$H_\mathbf{k}(z, t) = C_1 e^{|\mathbf{k}|z} + C_2 e^{-|\mathbf{k}|z} + \frac{1}{2|\mathbf{k}|}\int_z^0 dz_1 \chi_\mathbf{k}(z_1, t)\left[e^{|\mathbf{k}|(z-z_1)} - e^{-|\mathbf{k}|(z-z_1)}\right]. \tag{2.13}$$

As $z \to -\infty$

$$H_\mathbf{k}(z, t) \to e^{-|\mathbf{k}|z}\left\{C_2 - \frac{1}{2|\mathbf{k}|}\int_{-\infty}^0 dz_1 \chi_\mathbf{k}(z_1, t)\, e^{|\mathbf{k}|z_1}\right\} + \frac{1}{2|\mathbf{k}|}\int_z^0 dz_1 \chi_\mathbf{k}(z_1, t)\, e^{|\mathbf{k}|(z-z_1)}$$

and the condition on boundedness leads to the relation

$$C_2 = \frac{1}{2|\mathbf{k}|}\int_{-\infty}^0 dz_1 \chi_\mathbf{k}(z_1, t)\, e^{|\mathbf{k}|z_1}. \tag{2.14}$$

Using (2.14) it is possible to rewrite Eq. (2.13) in the form

$$H_\mathbf{k}(z, t) = C_1 e^{|\mathbf{k}|z} + \frac{1}{2|\mathbf{k}|}\int_z^0 dz_1 \chi_\mathbf{k}(z_1, t)\, e^{|\mathbf{k}|(z-z_1)} + \frac{1}{2|\mathbf{k}|}\int_{-\infty}^z dz_1 \chi_\mathbf{k}(z_1, t)\, e^{-|\mathbf{k}|(z-z_1)}.$$

The vertical component of Eq. (2.6) at z = 0 is used to determine the constant C_1:

$$\left.\frac{\partial H_\mathbf{k}}{\partial z}\right|_{z=0} = |\mathbf{k}|C_1 - \frac{1}{2}\int_{-\infty}^0 dz_1 \chi_\mathbf{k}(z_1, t)\, e^{|\mathbf{k}|z_1} = \left[-\left(\frac{\partial u_z^t}{\partial t} + R_z\right)_\mathbf{k}\right]_{z=0} \equiv -M_\mathbf{k}.$$

*The Hamiltonian variables describing even nonpotential motion of the free surface are given in [8].

We finally get the expression for H_k

$$H_{\mathbf{k}}(z,t)=\frac{1}{2|\mathbf{k}|}\int\limits_{z}^{0}dz_1\chi_{\mathbf{k}}(z_1,t)\left[e^{|\mathbf{k}|(z-z_1)}-e^{-|\mathbf{k}|(z-z_1)}\right]+\frac{1}{|\mathbf{k}|}\cosh|\mathbf{k}|z\int\limits_{-\infty}^{z}dz_1\chi_{\mathbf{k}}(z_1,t)e^{|\mathbf{k}|z_1}+\frac{M_{\mathbf{k}}}{|\mathbf{k}|}e^{|\mathbf{k}|z}. \quad (2.15)$$

Equations (2.1), (2.6), (2.8), and (2.10) along with (2.15) represent the original system to describe the interaction of gravity waves with hydrodynamic turbulence. We make the following assumptions before computing the respective matrix coefficients for the relationship. First, we assume that the characteristic vertical turbulence length L_z^t is much larger than the wavelength of surface waves $\lambda: L_z^t \gg \lambda$. Second, we assume that the relation $|\mathbf{u}^t| \ll v_\Phi$, is satisfied near the surface, which, as a rule, is observed in natural oceanic conditions (see e.g., [9]). It is well known [10, 11] that this assumption ensures weak interaction between waves and turbulence.

Equations (2.1), (2.10) are brought to the following form for the k-mode:

$$\psi_{\mathbf{k}}(t)=\frac{1}{(2\pi)^2}\int\psi(\mathbf{r}_\perp,t)\,e^{-i\mathbf{k}\mathbf{r}_\perp}d\mathbf{r}_\perp,\quad \eta_{\mathbf{k}}(t)=\frac{1}{(2\pi)^2}\int\eta(\mathbf{r}_\perp,t)\,e^{-i\mathbf{k}\mathbf{r}_\perp}d\mathbf{r}_\perp.$$

Complex amplitudes of normal modes $a_{\mathbf{k}}$, $a_{\mathbf{k}}^*$ are introduced in the standard manner [7]:

$$\eta_{\mathbf{k}}=\sqrt{\frac{|\mathbf{k}|}{2\omega_{\mathbf{k}}}}(a_{\mathbf{k}}+a_{-\mathbf{k}}^*),\quad \psi_{\mathbf{k}}=-i\sqrt{\frac{\omega_{\mathbf{k}}}{2|\mathbf{k}|}}(a_{\mathbf{k}}-a_{-\mathbf{k}}^*),$$

where $\omega_{\mathbf{k}}=\sqrt{g|\mathbf{k}|}$ is the dispersion law for gravity waves, and the equation of motion in the new variables has the form

$$\begin{aligned}\frac{\partial a_{\mathbf{k}}}{\partial t}+i\omega_{\mathbf{k}}a_{\mathbf{k}}=&-i\int\left[V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}a_{\mathbf{k}_1}a_{\mathbf{k}_2}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)+\right.\\&\left.+2u_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}a_{\mathbf{k}_1}a_{\mathbf{k}_2}^*\delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2)\right]d\mathbf{k}_1d\mathbf{k}_2-\\&-i\int V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}a_{\mathbf{k}_1}^*a_{\mathbf{k}_2}a_{\mathbf{k}_3}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3-\\&-i\int\left\{S^{\alpha}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\left[a_{\mathbf{k}_1}u^{\alpha}_{\mathbf{k}_2}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)+a_{\mathbf{k}_1}^*u^{\alpha}_{\mathbf{k}_2}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\right]+\right.\\&\left.+W^{\alpha\beta}_{\mathbf{k}}u^{*\alpha}_{\mathbf{k}_1}u^{*\beta}_{\mathbf{k}_2}\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\right\}d\mathbf{k}_1d\mathbf{k}_2.\end{aligned}\quad (2.16)$$

The following relations between $\psi_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}\equiv\frac{1}{(2\pi)^2}\int\varphi(\mathbf{r}_\perp,0,t)\,e^{-i\mathbf{k}\mathbf{r}_\perp}d\mathbf{r}_\perp$ has been used in the conversion of equation from (2.1), (2.10) to (2.16):

$$\begin{aligned}\varphi_{\mathbf{k}}=\psi_{\mathbf{k}}&+\int|\mathbf{k}_1|\psi_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,d\mathbf{k}_1d\mathbf{k}_2-\frac{1}{2}\int|\mathbf{k}_1|\,[\,|\mathbf{k}-\\&-\mathbf{k}_2|+|\mathbf{k}-\mathbf{k}_3|-|\mathbf{k}|\,]\,\psi_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\eta_{\mathbf{k}_3}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3\end{aligned}$$

with an accuracy to the order of $\sim a_{\mathbf{k}}^3$. The small parameter for the series expansion is $\xi = \sqrt{\frac{E}{\rho(\omega_{\mathbf{k}}/k)^2}}\ll 1$, where E is the surface wave energy. For the horizontal component of the turbulent velocity field $u^{\alpha}_{\mathbf{k}}(t)=\frac{1}{(2\pi)^2}\int u^t_\alpha(\mathbf{r}_\perp,\eta(\mathbf{r}_\perp,t),t)\,e^{-i\mathbf{k}\mathbf{r}_\perp}d\mathbf{r}_\perp\ (\alpha=x,y)$ it is possible to obtain the following equations in the k-mode* from Eq. (2.6) using Eq. (2.15):

$$\begin{aligned}\frac{\partial u^{\alpha}_{\mathbf{k}}}{\partial t}=&-\frac{i}{2}P^{\alpha\beta\gamma}_{\mathbf{k}}\int u^{*\beta}_{\mathbf{k}_1}u^{*\gamma}_{\mathbf{k}_2}\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d\mathbf{k}_1d\mathbf{k}_2-i\int Q^{\alpha\beta}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\times\\&\times\left(a_{\mathbf{k}_1}-a_{-\mathbf{k}_1}^*\right)u^{\beta}_{\mathbf{k}_2}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,d\mathbf{k}_1d\mathbf{k}_2.\end{aligned}\quad (2.17)$$

Terms of the type $S^n a^{n+1}u^\alpha$, $Q^n a^{n+1}u^\alpha$, $V^n a^{n+3}$ which are small when compared to ξ^n are omitted from Eqs. (2.16), (2.17) in further analysis.

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*The incompressibility condition $u^t_z(\mathbf{r}_\perp,z,t)=-\int\limits_{-\infty}^{z}\mathrm{div}_\perp\mathbf{u}^t_\perp(\mathbf{r}_\perp,z',t)\,dz'$ couples the vertical component of the turbulent velocity field to equation for horizontal components.

The coefficient $P_{\mathbf{k}}^{\alpha\beta\gamma}$ describes the interaction of turbulent fluctuations and is determined as follows:

$$P_{\mathbf{k}}^{\alpha\beta\gamma} = k_\beta \Delta_{\mathbf{k}}^{\alpha\gamma} + k_\gamma \Delta_{\mathbf{k}}^{\alpha\beta},\ \Delta_{\mathbf{k}}^{\alpha\beta} = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}.$$

Matrix coefficients $V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$, $U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ and $V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ describing the mutual interaction processes of surface waves were found in [7]. We have computed the coefficients $S_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\alpha}$, $W_{\mathbf{k}}^{\alpha\beta}$ and $Q_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\alpha\beta}$, describing the interaction of surface waves with the "horizontal" hydrodynamic turbulence. With these assumptions their explicit expressions have the form:

$$S_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\alpha} = \frac{1}{2}\sqrt{\frac{|\mathbf{k}||\mathbf{k}_1|}{\omega_{\mathbf{k}}\omega_{\mathbf{k}_1}}}\left[\frac{\omega_{\mathbf{k}}}{|\mathbf{k}|}k_\alpha + \frac{\omega_{\mathbf{k}_1}}{|\mathbf{k}_1|}k_{1\alpha}\right],\ W_{\mathbf{k}}^{\alpha\beta} = \sqrt{\frac{|\mathbf{k}|}{2\omega_{\mathbf{k}}}}\frac{k_\alpha k_\beta}{k^2},$$

$$Q_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\alpha\beta} = \sqrt{\frac{\omega_{\mathbf{k}_1}}{2|\mathbf{k}_1|}}\left[(\mathbf{k}_1\mathbf{k}_2)\,\delta_{\alpha\beta} - k_{1\beta}k_{2\alpha}\right],$$

where S describes the scattering of surfaces waves, and W and Q describe the processes of dissipation and generation of waves by turbulence.

3. Let us now proceed to the statistical description of nonlinear fields $u_{\mathbf{k}}$ and $u_{\mathbf{k}}^{\alpha}$. For this purpose the following averaged characteristics, viz., the correlator pairs are introduced in the $\mathbf{k}$-ω-mode

$$\langle u_{\mathbf{k}\omega}^{\alpha} u_{\mathbf{k}'\omega'}^{*\beta}\rangle = J_{\mathbf{k}\omega}^{\alpha\beta}\,\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega'),$$
$$\langle a_{\mathbf{k}\omega} a_{\mathbf{k}'\omega'}^{*}\rangle = n_{\mathbf{k}\omega}\,\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega') \tag{3.1}$$

and the Green's function

$$\left\langle \frac{\delta u_{\mathbf{k}\omega}^{\alpha}}{\delta F_{\mathbf{k}'\omega'}^{*\beta}} \right\rangle = G_{\mathbf{k}\omega}^{\alpha\beta}\,\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega'),$$
$$\left\langle \frac{\delta a_{\mathbf{k}\omega}}{\delta f_{\mathbf{k}'\omega'}^{*}} \right\rangle = g_{\mathbf{k}\omega}\,\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega').$$

The auxiliary function $G_{\mathbf{k}\omega}^{\alpha\beta}$ and $g_{\mathbf{k}}$ describe the response to the vortex and potential components of the velocity field to the external action of $\mathbf{F}_{\mathbf{k}\omega}^{\alpha\beta}$ and $f_{\mathbf{k}\omega}$, introduced in the right-hand side of the equations of motion (2.16), (2.17).

In view of homogeneous and isotropic turbulence in the horizontal plane, the spectral tensors $J_{\mathbf{k}\omega}^{\alpha\beta}$ and $G_{\mathbf{k}\omega}^{\alpha\beta}$ can be expressed in the form

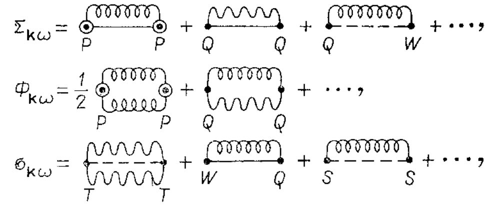
$$J_{\mathbf{k}\omega}^{\alpha\beta} = J_{\mathbf{k}\omega}\Delta_{\mathbf{k}}^{\alpha\beta},\ G_{\mathbf{k}\omega}^{\alpha\beta} = G_{\mathbf{k}\omega}\Delta_{\mathbf{k}}^{\alpha\beta}.$$

Dyson's equations can be obtained in the standard manner [6] for the mean characteristics (3.1)

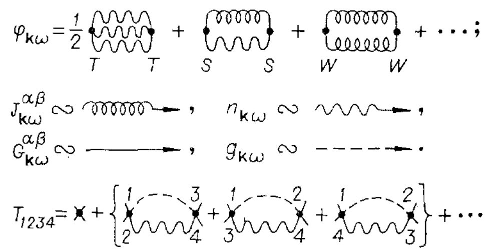
$$J_{\mathbf{k}\omega} = |G_{\mathbf{k}\omega}|^2\,\Phi_{\mathbf{k}\omega},\ n_{\mathbf{k}\omega} = |g_{\mathbf{k}\omega}|^2\hat{\varphi}_{\mathbf{k}\omega},$$
$$G_{\mathbf{k}\omega} = (\omega - \Sigma_{\mathbf{k}\omega})^{-1},\ g_{\mathbf{k}\omega} = (\omega - \omega_{\mathbf{k}} - \sigma_{\mathbf{k}\omega})^{-1}.$$

The first diagrams for $\Sigma_{\mathbf{k}\omega}$, $\Phi_{\mathbf{k}\omega}$ and $\sigma_{\mathbf{k}\omega}$, $\hat{\varphi}_{\mathbf{k}\omega}$, are shown in the figure where the corresponding correlator pairs and Green's function are also given.

The vertex $T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ appears in the diagrams for turbulent wave and is determined as follows*:



*In view of the dispersion law for gravity waves the triple wave interactions are not resonant; these processes contribute to quadruple wave matrix elements [12].



$$T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} - 2\frac{U_{-\mathbf{k}_2-\mathbf{k}_3\mathbf{k}_2\mathbf{k}_3}U_{-\mathbf{k}-\mathbf{k}_1\mathbf{k}\mathbf{k}_1}}{\omega_{\mathbf{k}}+\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}+\mathbf{k}_1}} -$$

$$-2\frac{V_{\mathbf{k}_2+\mathbf{k}_3\mathbf{k}_2\mathbf{k}_3}V_{\mathbf{k}+\mathbf{k}_1\mathbf{k}\mathbf{k}_1}}{\omega_{\mathbf{k}+\mathbf{k}_1}-\omega_{\mathbf{k}}-\omega_{\mathbf{k}_1}} - 2\frac{V_{\mathbf{k}\mathbf{k}_2\mathbf{k}-\mathbf{k}_2}V_{\mathbf{k}_3\mathbf{k}_1\mathbf{k}_3-\mathbf{k}_1}}{\omega_{\mathbf{k}_3-\mathbf{k}_1}+\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_3}} -$$

$$-2\frac{V_{\mathbf{k}_1\mathbf{k}_3\mathbf{k}_1-\mathbf{k}_3}V_{\mathbf{k}_2\mathbf{k}\mathbf{k}_2-\mathbf{k}}}{\omega_{\mathbf{k}_2-\mathbf{k}}+\omega_{\mathbf{k}}-\omega_{\mathbf{k}_2}} - 2\frac{V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_1-\mathbf{k}_2}V_{\mathbf{k}_3\mathbf{k}_1\mathbf{k}_3-\mathbf{k}_1}}{\omega_{\mathbf{k}_3-\mathbf{k}_1}+\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_3}} -$$

$$-2\frac{V_{\mathbf{k}\mathbf{k}_3\mathbf{k}-\mathbf{k}_3}V_{\mathbf{k}_2\mathbf{k}_1\mathbf{k}_2-\mathbf{k}_1}}{\omega_{\mathbf{k}_2-\mathbf{k}_1}+\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}}.$$

The kinetic equation serves as the basis for the statistical description of wave motion and for this purpose principal use is made of the weak nonlinearity, the presence of dispersion, and the assumption on the random nature of the phase of interacting waves. In the diagrammatic representation, the role of kinetic equation is played by the equation [13]

$$l_{\mathbf{k}\omega} = \operatorname{Im}(\sigma_{\mathbf{k}\omega}n_{\mathbf{k}\omega} + \hat{\varphi}_{\mathbf{k}\omega}g^*_{\mathbf{k}\omega}) = 0,$$

which is equivalent to the Dyson equation for $n_{\mathbf{k}\omega}$. In the case of weak turbulence it is possible to limit to diagrams of second order with respect to the vertices $T_{kk_1k_2k_3}$. Then the condition $l_{\mathbf{k}} \equiv \int l_{\mathbf{k}\omega}d\omega = 0$ leads to the stationary kinetic equation

$$0 = -\Gamma_{\mathbf{k}}n_{\mathbf{k}} + \pi\hat{\varphi}_{\mathbf{k}},$$

where

$$\Gamma_{\mathbf{k}} = -\operatorname{Im}\sigma_{\mathbf{k}\omega_{\mathbf{k}}},\ \hat{\varphi}_{\mathbf{k}} = \hat{\varphi}_{\mathbf{k}\omega_{\mathbf{k}}}.$$

It is possible to use an analogous equation for hydrodynamic turbulence

$$L_{\mathbf{k}\omega} = \operatorname{Im}(\Sigma_{\mathbf{k}\omega}J_{\mathbf{k}\omega} + \Phi_{\mathbf{k}\omega}G^*_{\mathbf{k}\omega}) = 0,$$

where the assumption on the nonrenormability of the vertex $P_{\mathbf{k}}^{\alpha\beta\gamma}$ corresponds to the direct interaction model [14]. However, such an approximation is not satisfactory for describing the properties of turbulence in the inertial interval. This is because the initial diagram series for $\Sigma_{\mathbf{k}\omega}$ and $\Phi_{\mathbf{k}\omega}$ possesses divergence of integrals in the region of small k, which physically stems from the effect of the transfer of small scale eddies by energy-carrying vortices. In order to describe the interaction of hydrodynamic "surface" waves it is therefore necessary, following [15], to use the equations

$$\tilde{L}_{\mathbf{k}\omega} = \operatorname{Im}(\tilde{\Sigma}_{\mathbf{k}\omega}\tilde{J}_{\mathbf{k}\omega} + \tilde{\Phi}_{\mathbf{k}\omega}\tilde{G}^*_{\mathbf{k}\omega}) = 0, \quad (3.2)$$

in which renormalization has been made to eliminate the kinematic effect of transfer. The terms in Eq. (3.2) are related to the original function as follows:

$$L_{\mathbf{k}\omega} = \langle\tilde{L}_{\mathbf{k}\omega-\mathbf{k}\mathbf{u}}\rangle_{\mathbf{u}},\ J_{\mathbf{k}\omega} = \langle\tilde{J}_{\mathbf{k}\omega-\mathbf{k}\mathbf{u}}\rangle_{\mathbf{u}},\quad G_{\mathbf{k}\omega} = \langle\tilde{G}_{\mathbf{k}\omega-\mathbf{k}\mathbf{u}}\rangle_{\mathbf{u}},\quad \Sigma_{\mathbf{k}\omega} = \langle\tilde{\Sigma}_{\mathbf{k}\omega-\mathbf{k}\mathbf{u}}\rangle_{\mathbf{u}},\ \Phi_{\mathbf{k}\omega} = \langle\tilde{\Phi}_{\mathbf{k}\omega-\mathbf{k}\mathbf{u}}\rangle_{\mathbf{u}},$$

where $\langle\ldots\rangle_{\mathbf{u}}$ represents averaging with respect to the random velocity field at the arbitrary point $\mathbf{r}$, t using Wyld's technique. As a result, the improved equation $\int\tilde{L}_{\mathbf{k}\omega}d\omega = 0$ in the two dimensional case,* as shown in [17], allows an exact solution with Kolmogorov values for the indices. Explicit expressions for Green's function and the correlator pair for the velocity field (for a constant energy flux) is written in the form

*The analysis of corresponding equations for 3-D turbulence is given in [16].

$$\tilde{G}_{\mathbf{k}\omega} = \omega^{-1} g\left(\frac{\omega L}{v_{\mathrm{T}} (kL)^{2/3}}\right), \; \tilde{J}_{\mathbf{k}\omega} = \frac{v_{\mathrm{T}}}{L^{1/3}} k^{-10/3} f\left(\frac{\omega L}{v_{\mathrm{T}} (kL)^{2/3}}\right). \quad (3.3)$$

Here v_T and L are respectively the chacteristic horizontal velocity and the scale for the energy containing segment of the turbulence spectrum.

4. One of the effects of the presence of hydrodynamic vortex turbulence is the attenuation of gravity waves. Attenuation occurs as a result of the direct absorption of surface energy by turbulence and also due to scattering. Consider now the first of these mechanisms.

In order to find the logarithmic decrement Γ_{dis} of gravity waves it is necessary to compute the contribution from the vertices W and Q to the imaginary part $\sigma_{\mathbf{k}\omega}$. Here it is sufficient to limit to the diagrams of the order WQ (see the diagram) since the diagrams containing W^2Q^2, $WQS^n{}_1$, are small compared to v_T/v_ϕ.

Analytical expression for Γ_{dis} in the present case has the form

$$\Gamma_{\mathrm{dis}} = -\operatorname{Im} \sigma_{\mathbf{k}\omega_{\mathbf{k}}} = -\operatorname{Im} \int W_{\mathbf{k}}^{\alpha\beta} Q_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}^{\delta\gamma} G_{\mathbf{k}_1\omega_1}^{\alpha\delta} J_{\mathbf{k}_2\omega_2}^{\beta\gamma} \times [\delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k})\, \delta(\omega_1 - \omega_2 + \omega_{\mathbf{k}}) - \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k})\, \delta(\omega_1 - \omega_2 - \omega_{\mathbf{k}})]\, d\mathbf{k}_1 d\mathbf{k}_2 d\omega_1 d\omega_2. \quad (4.1)$$

The basic contribution to the expression (4.1) comes from the integration in the region of scales k_1, $k_2 \sim k_T$, where the complex frequency $\omega_T \simeq (v_T/L)(k_T L)^{2/3}$ is of the order $\omega_{\mathbf{k}}$. Taking into consideration that the turbulent wave number k_T lies in the inertial interval $L^{-1} < k_T < L^{-1}Re^{3/4}$ (Re is the Reynolds number), or

$$\frac{v_{\mathrm{T}}^2}{gL} < kL < \frac{v_{\mathrm{T}}^2}{gL} \mathrm{Re}, \quad (4.2)$$

it is possible to estimate (4.1) in the form

$$\Gamma_{\mathrm{dis}} \simeq \frac{v_{\mathrm{T}}}{L} \left(\frac{v_{\mathrm{T}}}{v_\phi}\right)^2. \quad (4.3)$$

A comparison of Γ_{dis} with the logarithmic decrement $\Gamma_0 \simeq 2\nu k^2$ caused by molecular viscosity ν, shows that $\Gamma_{dis} \approx \Gamma_0$ when $k_0 \approx L^{-1} \frac{v_{\mathrm{T}}^2}{gL} \mathrm{Re}$, and, consequenly, the dissipation process associated with direct wave energy absorption by turbulence dominates in the entire interval (4.2).

For natural oceanic turbulence the decrement (4.3) is small since $v_T \ll v_\phi$. In the case of "artificial" turbulence caused by, e.g., a moving ship, wave dissipation can be appreciable (compare with [18]).

5. In the region $L^{-1} < k < L^{-1}Re^{3/4}$ is is necessary to consider the scattering of waves on turbulence. Firstly we find the logarithmic decrement of plane waves $\Gamma_{\mathbf{k}}$ due to the energy transfer from the given wave to the scattering field. The diagram series for $\sigma_{\mathbf{k}\omega}$ which describes the wave scattering on turbulence is determined by the vertices S (see diagram).

In the range $1 < kL < (gL/v_T^2)^{1/3}$, it is possible to limit to diagrams of the second order in vertices S. Then the damping is given by

$$\Gamma_{\mathbf{k}} = -\operatorname{Im} \sigma_{\mathbf{k}\omega_{\mathbf{k}}} = -\operatorname{Im} \int S_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\alpha} S_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}^{\beta} g_{\mathbf{k}_1\omega_1} J_{\mathbf{k}_2\omega_2}^{\alpha\beta} \delta(\mathbf{k} - \mathbf{k}_1 + \mathbf{k}_2)\, \delta(\omega_{\mathbf{k}} - \omega_1 - \omega_2)\, d\mathbf{k}_1 d\mathbf{k}_2 d\omega_1 d\omega_2. \quad (5.1)$$

Substituting Green's function in (5.1) and considering that $v_T/v_\phi \ll 1$, we get

$$\Gamma_{\mathbf{k}} \simeq \frac{\pi k^2}{2\left|\frac{d\omega_{\mathbf{k}}}{dk}\right|} \int_{L^{-1}}^{2k} dk' \sqrt{4 - \frac{k'^2}{k^2}} \int_{-\infty}^{\infty} d\omega' J_{k'\omega'}. \quad (5.2)$$

The energy containing vortices $k' \sim L^{-1}$ make major contribution to the Kolmogorov spectrum (3.3). Hence it follows from (5.2)

$$\Gamma_{\mathbf{k}} \approx k v_{\mathrm{T}} (kLv_{\mathrm{T}}/v_{\mathrm{gr}}) \simeq v_{\mathrm{T}}^2 g^{-1/2} k^{5/2} L, \quad (5.3)$$

where $v_{gr} = d\omega_k/dk$ is the group velocity of the surface waves. The expression obtained by us for Γ_k confirms the result obtained earlier in [10] without detailed computation of matrix elements. In scattering processes when the total energy of gravity waves is conserved, the quantity $\tau_{cor} = \Gamma_k^{-1}$ can be considered to be an estimate of the characteristic time decrement for the phase correlation coefficient of the wave field in the interval $1 < kL < (gL/v_T^2)^{1/3}$.

We qualitatively obtain (5.3) using similar arguments as in [4], considering the surface wave scattering on vortices of size k_T^{-1}. During one scattering event the Doppler shift in the frequency $\Delta\omega_k \approx kv_T(k_T)$ leads to a drop in the phase of gravity waves by an amount $\Delta\varphi \approx kv_T(k_T)/k_T v_{gr}$, where $v_T(k_T) \simeq v_T(k_T L)^{-1/3}$ is the "Kolmogorov" angular velocity of the vortex. $N \approx k_T v_{gr}\tau$ scatterings take place in time τ when the phase change increases $\sqrt{N}$ times due to the diffusive nature of the process. The corresponding time for the breakdown of phase correlations can be estimated from the condition $\Delta\varphi\sqrt{N} \approx 1$:

$$\tau^{-1} \approx \frac{v_T^2 g^{-1/2} k^{5/2} L}{(k_T L)^{5/3}}. \tag{5.4}$$

Since the major contribution to the scattering of plane waves is made by vortices with $k_T \sim L^{-1}$, then (5.4) corresponds to the result (5.3).

The case $kL > (gL/v_T^2)^{1/3}$ requires separate consideration since it is necessary to formally consider diagrams with four and more vertices S.

Let us study the growth in the intensity of wave packets with finite width Δk during scattering on turbulence, which is described by the nonstationary kinetic equation

$$\frac{1}{2}\dot{n}_{\mathbf{k}} = -\Gamma_{\mathbf{k}} n_{\mathbf{k}} + \pi\widehat{\varphi}_{\mathbf{k}\omega_{\mathbf{k}}}. \tag{5.5}$$

In the interval $1 < kL < (gL/v_T^2)^{1/3}$, using the diagram with vertices SS (see diagram) for $\widehat{\varphi}_{k\omega_{\mathbf{k}}}$ and taking into account Eq. (5.1) for Γ_k we get from (5.5)

$$\dot{n}_{\mathbf{k}} = \frac{\pi}{2}\int \frac{|\mathbf{k}||\mathbf{k}_1|}{\omega_{\mathbf{k}}\omega_{\mathbf{k}_1}}\left(\frac{\omega_{\mathbf{k}}}{|\mathbf{k}|}k_\alpha + \frac{\omega_{\mathbf{k}_1}}{|\mathbf{k}_1|}k_{1\alpha}\right)\left(\frac{\omega_{\mathbf{k}}}{|\mathbf{k}|}k_\beta + \frac{\omega_{\mathbf{k}_1}}{|\mathbf{k}_1|}k_{1\beta}\right) \times \tag{5.6}$$
$$\times J^{\alpha\beta}_{\mathbf{k}_2\omega_2}\left[n_{\mathbf{k}_1} - n_{\mathbf{k}}\right]\delta(\mathbf{k} - \mathbf{k}_1 + \mathbf{k}_2)\,\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} + \omega_2)\,d\mathbf{k}_1 d\mathbf{k}_2 d\omega_2.$$

If the nature of plane wave scattering is determined by the interaction with energy containing vortices leading to the scattering at angles of the order $1/kL \ll 1$, then for the wave packet of width $\Delta k > L^{-1}$, vortices from the inertial interval already plays an appreciable role, while the characteristic scattering angle $\Delta\theta_S \approx \Delta k/k$. The logarithmic decrement $\Gamma_k(\Delta k)$ corresponding to this can be found by noting that in this case the region with length scales $k_2 \sim k$ makes the major contribution to Γ_k in the integration with respect to k_2:

$$\tau_{\mathbf{k}}^{-1}(\Delta k) = \Gamma_{\mathbf{k}}(\Delta k) = \frac{v_T^2 k^2 L}{\left|\frac{d\omega_{\mathbf{k}}}{dk}\right|(\Delta k L)^{5/3}}. \tag{5.7}$$

It is possible to estimate from Eq. (5.7) the time for making the wave isotropic, which is determined by the last stage of scattering, when $\Delta k \sim k$:

$$\tau_{is}^{-1} \approx \frac{v_T^2 k^2 L}{\left|\frac{d\omega_{\mathbf{k}}}{dk}\right|(kL)^{5/3}} = \frac{\omega_{\mathbf{k}}}{(kL)^{2/3}}\left(\frac{v_T}{v_\Phi}\right)^2. \tag{5.8}$$

Consider now the inelastic scattering leading to the development of wave packets in terms of frequencies. Since the variation in frequency during scattering is small ($\Delta\omega_k/\omega_k \sim v_T/v_\phi \ll 1$), it is possible to use diffusion approximation for ω_k. Further in studying inelastic scattering we limit our attention to isotropic distributions. Then it follows from (5.6)

$$\frac{\partial n_k}{\partial t} = \frac{\partial}{\partial\omega_k} D_k \frac{\partial n_k}{\partial\omega_k}, \tag{5.9}$$

where

$$D_k = \frac{\pi k^2}{2} \int\limits_{L^{-1}}^{2k} dk' \sqrt{4 - \frac{k'^2}{k^2}} \int\limits_{-\infty}^{\infty} \omega'^2 J_{k'\omega'} d\omega'.$$

Equation (5.9) assumes that the characteristic time for isotropification (5.8) is much less than the diffusion time τ_{dif} determined by the coefficient D_k. Since we have the following estimate for D_k on Kolmogorov spectrum (3.3)

$$D_k \approx \frac{\omega_k^3}{(kL)^{4/3}} \left(\frac{v_T}{v_\Phi}\right)^4 \frac{v_\Phi}{v_{gr}},$$

then the characteristic time τ_{dif} given by

$$\tau_{dif}^{-1} \approx \frac{D_k}{\omega_k^2} \simeq \frac{\omega_k}{(kL)^{4/3}} \left(\frac{v_T}{v_\Phi}\right)^4 \frac{v_\Phi}{v_{gr}},$$

is $(kL)^{2/3}(v_\phi/v_T)^2$ times more than τ_{is}, which also substantiates the initial assumption.

6. In conclusion we discuss the question of the limits of applicability of kinetic equation for the description of mutual interaction of gravity waves in a turbulent medium. First of all we note that the mutual interaction of surface waves in a turbulent medium is possible only when the characteristic time for nonlinear wave interaction τ_{int} is less than the characteristic time Γ_{dis}^{-1}. Using the following expression from [19] for τ_{int}

$$\tau_{int}^{-1} \approx \omega_k \left(\frac{E}{\rho \left(\frac{\omega_k}{k}\right)^2}\right)^2,$$

we find from the inequality $\tau_{int}^{-1} > \Gamma_{dis}$

$$E > \rho v_T^2 \left(\frac{v_\Phi}{kLv_T}\right)^{1/2}. \tag{6.1}$$

Upon satisfaction of condition (6.1), the nonlinear interaction of the waves becomes defined; this interaction can be described by the kinetic equation (19) using diagram series $\sigma_{k\omega}$, $\varphi_{k\omega}$ with vertices TT (see figure).

This approximation will be correct if the necessary condition on the width of Δk bundles of the surface disturbance [13] is satisfied:

$$\tau_d^{-1} = \omega_k'' (\Delta k)^2 \gg \tau_{int}^{-1}. \tag{6.2}$$

Expression (6.2) follows formally from the condition that the diagrams renormalizing the vertex $T_{kk_1k_2k_3}$ (see diagram) are small.

Consequently, in this case an additional condition besides (6.2) appears for the applicability of the kinetic equation:

$$\tau_{int}^{-1} < \Gamma_k, \quad 1 < kL < \left(\frac{gL}{v_T^2}\right)^{1/3}. \tag{6.3}$$

Expressing (6.3) in terms of energy density of surface waves, we get

$$E < E_{cr} = \rho v_T^2 \left(\frac{v_\Phi kL}{v_{rp}}\right)^{1/2} \frac{v_\Phi}{v_T}, \quad 1 < kL < \left(\frac{gL}{v_T^2}\right)^{1/3}. \tag{6.4}$$

It is significant that the width of the wave packet does not enter the inequality (6.4) and in the scattering process the phase becomes random even for a solitary wave.

In fulfilling the criteria (6.1)-(6.4) the following Zakharov—Filonenko spectrum [19] will be established as a result of nonlinear mutual interaction of gravity waves

$$E_k \sim k^{-5/2}, \tag{6.5}$$

Here the role of turbulence will lead to the effect of isotropification of wave packets in the characteristic time τ_{is} (see (5.8)). For sufficiently large wave amplitudes, strongly nonlinear processes come into play and the theory of weak turbulence becomes inapplicable. The breaking of waves at the crest is such a process in the gravitational field. As a result of breaking in this range of wave numbers the spectral distribution (6.5), apparently, should transform to the Phillips spectrum [20]

$$E_k \sim k^{-3}.$$

In actual conditions the above picture may be significantly complicated due to the influence of wind, interaction of gravity waves with capillaries and so on.

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